Wednesday $10^h45 - 12^h15$ + $16^h15 - 18^h15$
Christophe Mourougane

Deformation of complex manifolds

We may classify compact complex curves via the genus.

Chapter I. Analytic families of complex manifolds

I) Reminders on complex manifold

Complex manifolds, Morrow, Kodaira.

Four natural operations on complex manifolds

① • product, disjoint union

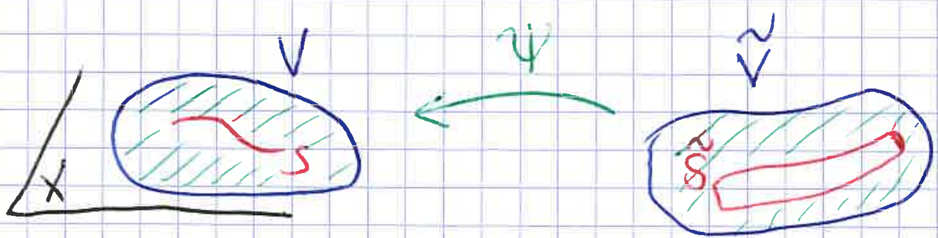
② • surgery

[Lee: 100]

If X is a complex manifold, S a closed submanifold, V a neighb. of S in X, $\tilde{V}$ a complex mfd, $\tilde{S}$ a closed submfd of $\tilde{V}$ and $\psi : \tilde{V} - \tilde{S} \to V - S$ biholomorphism, then we can endow

S compact? See hypothesis in [Morrow-Kodaira:15] + cf. p ⑩

$$(X-S) \underset{\psi}{\cup} \tilde{V} \overset{\text{def}}{:=} (X-S) \sqcup \tilde{V} \Big/ x \sim \tilde{v} \text{ iff } x \in V - S,\ \tilde{v} \in \tilde{V} - \tilde{S},\ \psi(\tilde{v}) = x$$



with a structure of complex manifold.

Proof works with S and $\tilde{S}$ as mere closed subsets...

Example: Blow-ups

If X is a (complex) ~~mfd~~ surface, $S = \{p\}$ a point of X,

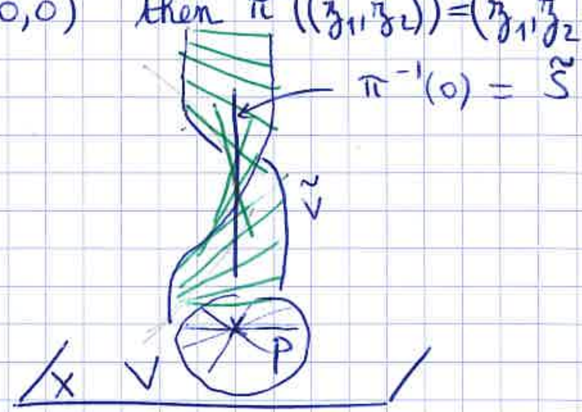
$$V = \{ (z_1, z_2) \in V : |z_1| < \varepsilon, |z_2| < \varepsilon \}$$

in a coordinate chart centered à p

$\tilde{V} \subset V \times \mathbb{P}^1$ defined by

$$\tilde{V} := \left\{ (z_1, z_2, [Z_1 : Z_2]) : \begin{vmatrix} z_1 & Z_1 \\ z_2 & Z_2 \end{vmatrix} = 0 \right\} \xrightarrow{\pi} V$$

If $(z_1, z_2) \neq (0,0)$ then $\pi^{-1}((z_1,z_2)) = (z_1, z_2, [z_1 : z_2])$



$$\begin{array}{c} \tilde{V} \\ \downarrow \pi \\ V \end{array}$$

$$\psi = \pi_{|\tilde{V} - \pi^{-1}(0)} : \tilde{V} - \tilde{S} \longrightarrow V - \{p\}.$$

$$\begin{array}{c} X - S \\ \downarrow id \\ X - S \end{array} \quad \text{glue to give} \quad \begin{array}{c} \tilde{X} = (X - S) \cup_{\psi} \tilde{V} \\ \downarrow \\ X \end{array}$$

In general, if Y is a submanifold of X, we can blow it up to get $Bl_Y(X) = \tilde{X}$ with a map $p : \tilde{X} \to X$ such that $p : \tilde{X} - p^{-1}(Y) \to X \setminus Y$ is a biholom. $p^{-1}(Y) \simeq P(N_{Y/X})$

projectivization

normal bundle of Y in X.

③ • The construction of sub-manifolds : relies on the <u>local holom. inversion theorem</u> : let $U \ni z_0$ be an open set in $\mathbb{C}^n$, and $\varphi : U \to \mathbb{C}^n$ a holomorphic map such that $\mathrm{Jac}(\varphi)_{z_0} \neq 0$.

Then, there is an open ~~set~~ neighb. V of z_0 in U such that $\varphi_{|V} : V \longrightarrow \varphi(V)$ is a biholomorphism.

[Griffith-Harris : 19]

<u>Proof</u> From differentiable point of view $\varphi_{\mathbb{R}} : U_{\mathbb{R}} \to \mathbb{R}^{2n}$.

X Jacobien

$\mathrm{Jac}\, \varphi_{\mathbb{R}} = |\mathrm{Jac}\, \varphi|^2 > 0$ in a neighb. of z_0.

By the local inversion theorem,

$$\exists V \in \mathcal{V}_{z_0}(U) \text{ s.t. } \varphi_{\mathbb{R}} : V \longrightarrow \varphi(V) \text{ diffeo}$$
$$(z_1 \dots z_n) \longmapsto (w_1 = \varphi_1(z_1 \dots z_n), w_2 = \varphi_2(\dots) \dots)$$

$$\psi = \varphi_{\mathbb{R}}^{-1} : \varphi(V) \longrightarrow V$$
$$(w_1, \dots, w_n) \longmapsto (\psi_1(\ \ ), \psi_2(\ \ ) \dots)$$

$$z_k = \psi_k(\varphi_1(z_1 \dots z_n), \dots)$$

$\forall l$ $\dfrac{\partial}{\partial \bar{z}_l}$

$$0 \underset{\text{chain rule}}{=} \sum \frac{\partial \psi_k}{\partial w_i} \frac{\partial \varphi_i}{\partial \bar{z}_l} + \sum \frac{\partial \psi_k}{\partial \bar{w}_i} \frac{\partial \bar{\varphi}_i}{\partial \bar{z}_l} = \sum \frac{\partial \psi_k}{\partial \bar{w}_i} \overline{\frac{\partial \varphi_i}{\partial z_l}}$$

But $\det\left(\overline{\dfrac{\partial \varphi_i}{\partial z_l}}\right) = \overline{\mathrm{Jac}(\varphi)} \neq 0$.

Hence $\forall l$ $\forall k$ $\dfrac{\partial \psi_k}{\partial \bar{w}_i} = 0 \Rightarrow$ <u>ψ is holomorphic</u>.

Hence $\varphi_{|V}$ is a biholom.

④ • Quotient : if X is a complex manifold, G a group acting by bihdom.
($G \to \mathrm{Biholo}(X)$ group homo) fixed point free ($\forall g \in G - \{e_G\}$ $\forall x \in X$ $g \cdot x \neq x$)
properly discontinuously ($\forall K_1, K_2$ compact in X, $\# \{ g \in G : g(K_1) \cap K_2 \neq \emptyset \}$ finite) then $X/_G$ has a natural structure of a complex mfd, such that $X \xrightarrow{p} X/_G$ is étale

($\forall x \in X$, $\exists V \in \mathcal{V}_X(x)$ s.t. $p_{|V} : V \to p(V)$ is biholomorphic).

This may not be a covering (the number of preimages may vary).

Proof [Morrow, Kodaira ; 12]

II Analytic / Holomorphic families of compact complex manifolds

Def A holomorphic family of compact complex manifolds is a map $\pi : \mathcal{X} \longrightarrow T$ between two complex manifolds such that (T connected)

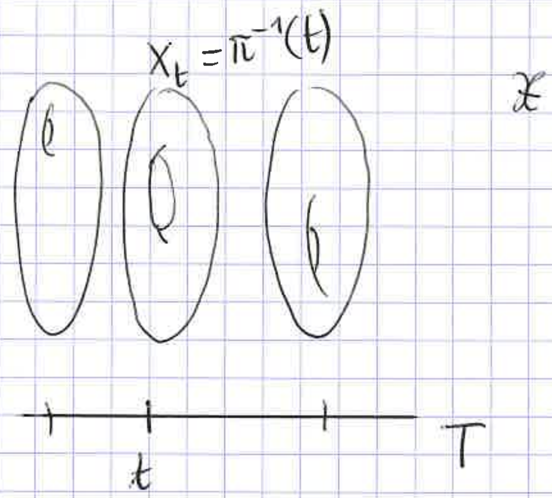
[Kodaira, Morrow] demandent seulement que les fibres soient compactes

(i) π is holomorphic

(ii) π is proper (i.e. $\forall K$ compact in T $\pi^{-1}(K)$ is compact).

(iii) π is submersive (i.e. $\forall x \in \mathcal{X}$, $d\pi_x : T_x\mathcal{X} \longrightarrow T_{\pi(x)}T$ is surjective).

Je pense que la surjectivité de π est demandée implicitement : $\forall t \in T$, X_t variété compacte.



Remark

$\forall t \in T$, $X_t = \pi^{-1}(t)$ the fiber over t is a compact manifold of $\mathcal{X}$ (by (i),(ii) and (iii)).

We will see that

Ehresmann
- the differentiable structure of the fibers X_t does not change (as long as T is connected)

Fischer-Grauert
- the holomorphic structure of the X_t may vary

La preuve n'est pas donnée.

1947 — Sur la théorie des espaces fibrés

cf. Oeuvres complètes p. 159 → p.159 (pdf 185)

Comptes rendus Ac. Sc. Paris, 224, 1947, p1611. MR0020774

~~Texte complet sur Gallica.~~ p.445 Gallica ←

Theorem (Ehresmann) $\boxed{\text{Extrait}}$ || /22/ des Oeuvres complètes, p. 326 (pdf 352)

Sur les esp. fibrés différentiables

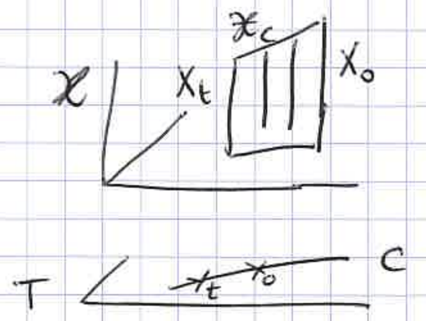
If T is a connected complex manifold, $\pi : \mathcal{X} \longrightarrow T$ a holomorphic family of complex manifolds, then all the fibers $X_t = \pi^{-1}(t)$ are diffeomorphic.

[Morrow, Kodaira: 20] ~~X~~ cf. p ⑨

Proof

$$\begin{array}{ccccc} & & \pi^{-1}(c) & & \\ & & \| & & \\ \mathcal{X} & \supset & \mathcal{X}_c & \supset & X_0 \\ \downarrow & & & & \downarrow \\ T & \supset & C & \ni & 0 \end{array}$$

real curve



We will show that for t close to 0, X_t is diffeom. to X_0.

$\mathcal{X}_c$ is a differentiable manifold. Choose a local diff. coord. $t : C \rightarrow \mathbb{R}$ centered at 0.

$\pi_{|\mathcal{X}_c} : \mathcal{X}_c \longrightarrow C$ is submersive.

Hence, by the local inversion theorem, there is a finite* open covering of a neighb. of X_0 in $\mathcal{X}_c$ with differentiable coordinate charts

* (compacity)

$$\varphi^\alpha : U^\alpha \longrightarrow V^\alpha \subset \mathbb{R}^{2n+1}$$

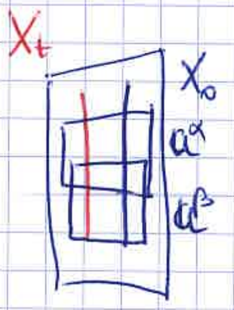
→ (cela fonctionne en toute dimension ; pas seulement paire)

$$p \longmapsto (x_1^\alpha(p), \dots, x_{2n}^\alpha(p), \tau^\alpha(p))$$

such that

$$
\begin{array}{ccc}
\mathcal{X}_c \supset U^\alpha & \xleftarrow{(\varphi^\alpha)^{-1}} & V^\alpha \ni (x_1^\alpha \dots x_{2n}^\alpha, \tau^\alpha) \\
\pi \downarrow & & \downarrow \\
C \xrightarrow{t} \mathbb{R} & & \tau^\alpha = t \circ \pi \circ (\varphi^\alpha)^{-1}(x_1^\alpha, \dots, x_{2n}^\alpha, \tau^\alpha)
\end{array}
$$

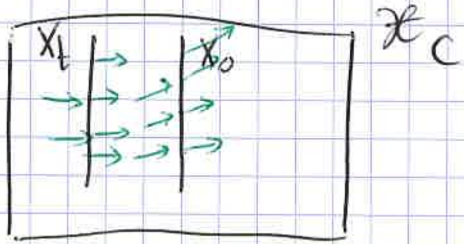
such that
(see [MK:20] for the embedded one γ)

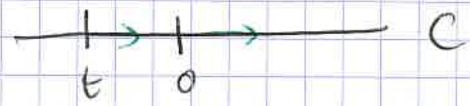


As $\mathcal{X}_c$ is paracompact, there is a partition of unity $(\rho^\alpha)_\alpha$ subject to the covering $(U^\alpha)_\alpha$
Consider the vector field

$$v \overset{\text{def}}{:=} \sum_\alpha \rho^\alpha \frac{\partial}{\partial \tau^\alpha}$$ is a global vector field

on $\mathcal{X}_c$ such that $d\pi \cdot v = \frac{\partial}{\partial t}$.





Consider the flow, the solution of

$$
(E) \quad \begin{cases} \dot{X} \overset{\text{def}}{:=} \dfrac{dX}{dt} = v \\ X(0) = Id_{X_0} \end{cases}
$$

On U^α

$$v = \sum_{i=1}^{2n} v_i^\alpha \frac{\partial}{\partial x_i^\alpha} + \frac{\partial}{\partial z^\alpha}$$

$$(E^\alpha) \quad \begin{cases} \dot{X}_i^\alpha = v_i^\alpha \\ X_i^\alpha(0) = x_i^\alpha \end{cases}$$

There are local solutions $X_i^\alpha(t)$.

Consider the coordinates change on $U^\alpha \cap U^\beta$

$$\begin{cases} x_i^\beta = f_i^{\beta\alpha}(x_1^\alpha, \dots, x_{2n}^\alpha, \tau^\alpha) \\ \tau^\beta = \tau^\alpha \end{cases}$$

$$\frac{\partial}{\partial x_i^\alpha} = \sum_k \frac{\partial f_k^{\beta\alpha}}{\partial x_i^\alpha} \frac{\partial}{\partial x_k^\beta}$$

$$\frac{\partial}{\partial z^\alpha} = \sum_k \frac{\partial f_k^{\beta\alpha}}{\partial \tau^\alpha} \frac{\partial}{\partial x_k^\beta} + \frac{\partial}{\partial \tau^\beta}$$

$$v = \sum_i v_i^\alpha \frac{\partial}{\partial x_i^\alpha} + \frac{\partial}{\partial \tau^\alpha}$$

$$= \sum_i v_i^\alpha \left(\sum_k \frac{\partial f_k^{\beta\alpha}}{\partial x_i^\alpha} \frac{\partial}{\partial x_k^\beta} \right) + \sum_k \frac{\partial f_k^{\alpha\beta}}{\partial z^\alpha} \frac{\partial}{\partial x_k^\beta} + \frac{\partial}{\partial \tau^\beta}$$

$$v_k^\beta = \sum_i v_i^\alpha \frac{\partial f_k^{\beta\alpha}}{\partial x_i^\alpha} + \frac{\partial f_k^{\alpha\beta}}{\partial z^\alpha}$$

$$\dot{X}_k^\beta = v_k^\beta = \sum_i \dot{x}_i^\alpha \frac{\partial f_k^{\beta\alpha}}{\partial x_i^\alpha} + \frac{\partial f_k^{\alpha\beta}}{\partial z^\alpha}$$

$$= \frac{d}{dt} f_k^{\beta\alpha}(X_1^\alpha, \dots, X_{2n}^\alpha, \tau^\alpha)$$

$$X_k^\beta(0) = f_k^{\beta\alpha}(X_1^\alpha(0), \dots, X_{2n}^\alpha(0), 0)$$

By unicity of the flow,

By unicity of the flow, $X_k^{\beta}(t) = f_k^{\beta\alpha}(X_1^{\alpha}(t), \dots, X_{2m}^{\alpha}(t), t)$

Hence,
$$\begin{aligned} X_0 &\longrightarrow X_1 \\ (x_i^{\alpha}) &\longmapsto (X_i^{\alpha}(t)) \end{aligned}$$
gives a diffeomorphism from X_0 to X_t $\square$

$(t, x_i^{\alpha}) \longmapsto (X_i^{\alpha}(t)) \quad \Rightarrow$ fiber bundle

III Examples

1) Family of elliptic curves

$\mathbb{H} = \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$

The group $\mathbb{Z}^2$ acts on $\mathbb{H} \times \mathbb{C}$ by biholomorphisms.

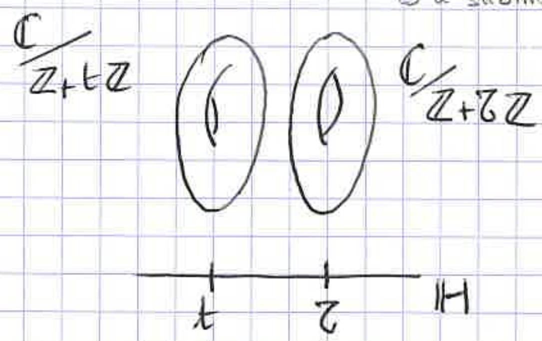
$$\mathbb{Z}^2 \times \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{H} \times \mathbb{C}$$

$$((m,n), t, z) \longmapsto (t, z + m + nt)$$

The action is fixed point free and properly discontinuous.

The quotient $\mathbb{H} \times \mathbb{C} / \mathbb{Z}^2 \overset{\text{def}}{=}: \mathcal{E}$ is a complex manifold and has a map $\mathcal{E} \xrightarrow{p_1} \mathbb{H}$, holomorphic, proper and submersive*. It is a holomorphic family of elliptic curves.

* quotient map is étale + composition of submersion is a submersion + $pr_1 = p_1 \circ q$



Questions

1) Are E_t and E_τ biholomorphic?
2) Can we deform $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ into other complex curves?
3) Can we deform $\mathbb{P}^1$?

$\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(0) =$ trivial line bundle over $\mathbb{P}^1$

2) Hirzebruch surfaces

a) Algebraic definition

$\mathbb{P}^1$ complex projective line. $m \in \mathbb{N}$

Consider the algebraic vector bundle $E_m = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m) \to \mathbb{P}^1$

$F_m \overset{\text{def}}{=} \mathbb{P}(E_m)$ the variety of rank one quotients of E_m.

where $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$

V $\mathbb{C}$-vector space

see Grassmaniann

$\mathbb{P}(V)$ is the variety of $\begin{cases} \text{rank one quotients of } V \\ \text{hyperplanes in } V \\ \text{lines in } V^* \end{cases}$

$P(V) = \mathbb{P}(V^*)$ serait la notation classique.

It is an algebraic manifold. (en dimension $2 = 1+1$, la dimension reste la même)

It has a ~~tangent~~ tautological quotient algebraic line bundle denoted by $\mathcal{O}_{\mathbb{P}(V)}(1)$:

see [Hartshorne]

Griffiths-Harris $\hookrightarrow k = \mathbb{C}$

Déf 1

For $q \in \mathbb{P}(V)$, $q : V \to L$ ("rank one quotient")

$$\left(\mathcal{O}_{\mathbb{P}(V)}(1)\right)_q = L \quad \text{(quotient)}$$

~~Connaissant L, comment retrouver q?~~

$\mathbb{P}(V) \times V \longrightarrow \mathcal{O}_{\mathbb{P}(V)}(1)$ C'est un quotient du fibré trivial

$(q, v) \longmapsto q(v)$

trivial vector bundle

See Picard group

$$\mathcal{O}_{\mathbb{P}(V)}(m) := \underbrace{\mathcal{O}_{\mathbb{P}(V)}(1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}(V)}(1)}_{m \text{ times}} \quad (\text{here } m \in \mathbb{N})$$

For $m \in \mathbb{Z}_-$:

$$\mathcal{O}_{\mathbb{P}(V)}(m) = \left(\mathcal{O}_{\mathbb{P}(V)}(-m)\right)^*$$

justifier cette définition

Def 2

V is a $\mathbb{C}$-vector space

$P(V)$ the variety of lines in V

$\mathcal{O}_{P(V)}(-1)$ is the tautological line bundle on $P(V)$ whose fiber at $l \in P(V)$ is simply $l \subset V$.

$$\mathcal{O}_{P(V)}(-1) \hookrightarrow P(V) \times V \quad \rightsquigarrow \text{sous-fibré du fibré trivial}$$
$$v_q \longmapsto (q, v)$$

- Choose a base $(v^0, \dots, v^n)$ of V.

This gives homogeneous coordinates $[Z_0 : \dots : Z_n]$ on $P(V)$.

On the open set $U^\alpha = (Z_\alpha \neq 0) \subset P(V)$ we get holomorphic coordinates by setting $z_i^\alpha := \frac{Z_i}{Z_\alpha}$, $i \neq \alpha$.

A frame for $\mathcal{O}_{P(V)}(-1)$ on U^α is a non-vanishing section

"repère" d'un fibré

$$(z_i^\alpha)_{\substack{i=0,\dots,n \\ i \neq \alpha}} \longmapsto \left(v^\alpha + \sum_{\substack{i=0 \\ i\neq\alpha}}^{n} z_i^\alpha v^i\right) \in \mathbb{C}\Big(\sum_{i=0}^{n} Z_i v^i\Big) = [Z_0 : \dots Z_n]$$

This gives a (local) trivialization of $\mathcal{O}_{P(V)}(-1)$ on U^α :

$$U^\alpha \times \mathbb{C} \longrightarrow \mathcal{O}_{P(V)}(-1)|_{U^\alpha}$$
$$((z_i^\alpha), \lambda) \longmapsto \lambda\Big(v^\alpha + \sum_{\substack{i=0 \\ i\neq\alpha}}^{n} z_i^\alpha v^i\Big)$$

$$\left(\begin{aligned} \lambda(v^\alpha + \sum z_i^\alpha v^i) &= \lambda(v^\alpha + \sum \frac{Z_i}{Z_\alpha} v^i) \\ &= \lambda \underbrace{\frac{Z_\beta}{Z_\alpha}}_{z_\beta^\alpha} \Big(v^\beta + \sum_{j\neq\beta} z_j^\beta v^j\Big) \end{aligned} \right)$$

$$(U^\alpha \cap U^\beta) \times \mathbb{C} \longrightarrow (U^\alpha \cap U^\beta) \times \mathbb{C}$$
$$(z_0^\alpha, \dots, z_n^\alpha, \lambda) \longmapsto (z_0^\alpha, \dots, z_n^\alpha, \lambda z_\beta^\alpha)$$

For $m \in \mathbb{Z}$, the line bundle $\mathcal{O}_{\mathbb{P}(V)}(m)$ have change of ~~coordinates~~ trivialisations given by

proof ?

$$\begin{aligned} \tau_{\alpha\beta} : \mathcal{U}^\alpha \cap \mathcal{U}^\beta \times \mathbb{C} &\longrightarrow \mathcal{U}^\alpha \cap \mathcal{U}^\beta \times \mathbb{C} \\ \lambda &\longmapsto \lambda \left(z^\alpha_\beta\right)^{-m} \end{aligned}$$

This fulfills the cocycle relation

$$\tau_{\beta\gamma} \circ \tau_{\alpha\beta} = \tau_{\alpha\gamma} \text{ on } \mathcal{U}^\alpha \cap \mathcal{U}^\beta \cap \mathcal{U}^\gamma.$$

Rq Ici, l'atlas standard de $\mathbb{P}^1$ trivialise le fibré, ce qui n'est pas évident à priori.

Link between the two constructions

Let V be a finite dimensional $\mathbb{C}$-v.s.

$$\begin{aligned} \mathbb{P}(V) &\longrightarrow \mathbb{P}(V^*) \\ (q : V \longrightarrow L) &\longmapsto \mathbb{C} \cdot \left(\text{~~an equation u for the hyperplane $\operatorname{Ker} q$~~} \right) \end{aligned}$$

a linear form u with kernel $\ker q$.

$$\mathcal{O}_{\mathbb{P}(V)}(-1)$$

$$L^* \overset{\|}{=} \left(V/_{\ker q}\right)^* \xrightarrow[\text{naturally isom}]{\sim} \mathcal{O}_{\mathbb{P}(V^*)}(-1)_{\mathbb{C}u} = \mathbb{C}u$$

$$\mathcal{O}_{\mathbb{P}(V^*)}(1) = (\mathbb{C}u)^*$$

je pense

$$\left(V/_{\ker q}\right)^* = \{ f_u \in V^* : f^u_{|\ker q} = 0 \} \quad \text{(factorization thm)}.$$

$$= \mathbb{C}u.$$

Cela donne un isomorphisme de fibrés : ...

$$\mathcal{O}_{\mathbb{P}^1}(0) = \begin{matrix} \mathbb{P}^1 \times \mathbb{C} \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$$

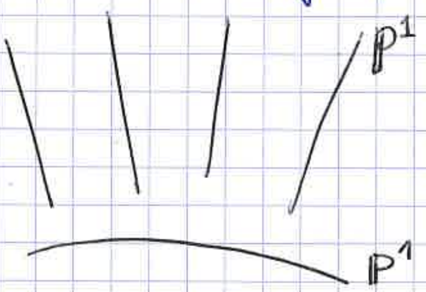
Back to the algebraic definition.

$$E_m \overset{\text{def}}{:=} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m) \longrightarrow \mathbb{P}^1$$

$F_m \overset{\text{def}}{:=} \mathbb{P}(E_m) \to \mathbb{P}^1$ the variety of rank one quotient of E_m.

Hirzebruch surface

$\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$



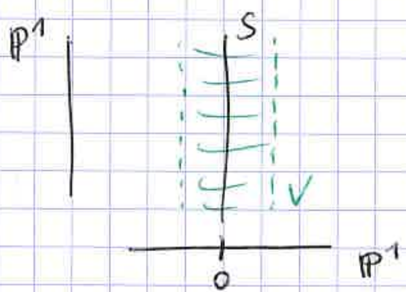
[Morrow-Kod:15] b) ~~By surjery~~

Consider $\mathbb{P}^1$ with homogeneous coordinates $[Z_0 : Z_1]$, and holomorphic coordinates $z = \frac{Z_0}{Z_1}$ on $U^1 = (Z_1 \neq 0)$.

Denote $0 = [0:1] \in U^1$ and $\infty = [1:0] \in U^0$, $\notin U^1$.

Denote $X = \mathbb{P}^1_z \times \mathbb{P}^1_\zeta$

$S = (Z_0 = 0) \subset X$ $\qquad V = (|z| < \varepsilon) \subset X$



$\hat{X}$ copy of X?

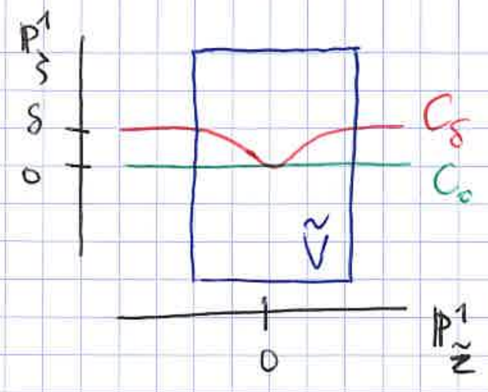
$\tilde{S} = (\tilde{Z}_0 = 0) \subset \tilde{X}$ $\qquad \tilde{V} = (|\tilde{z}| < \varepsilon) \subset \tilde{X}$

$$\psi : \tilde{V} - \tilde{S} \longrightarrow V - S$$

$$(\tilde{z}, \tilde{\zeta}) \longmapsto \left(z = \tilde{z},\ \zeta = \frac{\tilde{\zeta}}{\tilde{z}^m}\right) \qquad m \in \mathbb{Z}$$

is a biholomorphism.

$$F_m = (X - S) \underset{\psi}{\cup} \tilde{V}$$



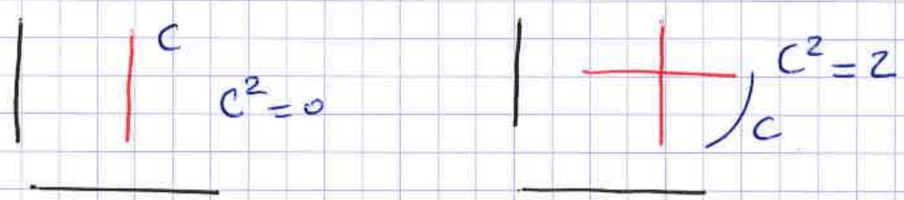
The curve given by $\left\{ \begin{array}{l} (\xi = \delta) \text{ on } X - S \\ (\tilde{\xi} = \delta \tilde{z}^m) \text{ on } \tilde{V} \end{array} \right\}$

glue on F_m.

The intersection product on F_m $C_\delta \cdot C_0 = m = C_0^2$

(C_δ small deformation of C_0)

But the intersection form on $\mathbb{P}^1 \times \mathbb{P}^1$ is even
(i.e. $\forall C$ curve in $\mathbb{P}^1 \times \mathbb{P}^1$, C^2 is even)



[This shows that for m odd, F_m is not homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 = F_0$.

c) Definition by an atlas

Consider $\mathbb{P}^1_{[Z_1:Z_2]}$ with the two coordinates charts U^1, U^2,

$$z_1 = \frac{Z_1}{Z_2} \text{ on } U^2 = (Z_2 \neq 0)$$

$$z_2 = \frac{Z_2}{Z_1} \text{ on } U^1 = (Z_1 \neq 0).$$

$U^2 \times \mathbb{P}^1_{\zeta_1}$ $\quad (z_1, \zeta_1)$

$U^1 \times \mathbb{P}^1_{\zeta_2}$ $\quad (z_2, \zeta_2)$

On $(U^2 \cap U^1) \times \mathbb{P}^1$, the coordinate change for F_m is defined to be

$$z_1 = f_1^{12}(z_2, \zeta_2) = \frac{1}{z_2}$$

$$\zeta_1 = f_2^{12}(z_2, \zeta_2) = z^m \zeta_2 \quad \left(\text{coordinate change for } \mathcal{O}_{\mathbb{P}^1}(m)\right)$$

The surface F_m has a map to $\mathbb{P}^1$ defined by

$$(z_1, \zeta_1) \longmapsto z_1 \text{ on } U^2 \times \mathbb{P}^1$$

$$(z_2, \zeta) \longmapsto z_2 \text{ on } U^1 \times \mathbb{P}^1.$$

The higher m is, the more twist we get.

22 January 2020

(à rendre la semaine prochaine)
Correction p. 33.

Exercise 1 (Link between algebraic definition a) and def c) by an atlas)

Let V be a complex vector space of rank 2.

Let $F_m = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$

Choose a base (e_1, e_2) of V

1) Find and atlas of F_m

2) Compare with construction of F_m by

$U_1 \times \mathbb{P}^1 \qquad (z_1, \zeta_1)$

$U_2 \times \mathbb{P}^1 \qquad (z_2, \zeta_2)$

In fact, this is the patching of $U_1 \times W$ with $U_2 \times W$.

patched by $z_1 = \frac{1}{z_2}$ and $\zeta = z_2^m \zeta_2$, where

$\mathbb{C}^2$? (vector).

$(\vec{i}, \vec{j})$ basis of $\mathbb{C}^2$

$[X, Y]$ homogeneous coordinates of $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^1)$

$z_1 := \frac{X}{Y}$ on $(Y \neq 0) = U_1$

$z_2 := \frac{Y}{X}$ on $(X \neq 0) = U_2$

and

(v_1, v_2) basis of $\mathbb{C}^2$

$[V_1 : V_2]$ homogeneous coordinates on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$

$\zeta = \frac{V_1}{V_2}$ on $W = (V_2 \neq 0)$.

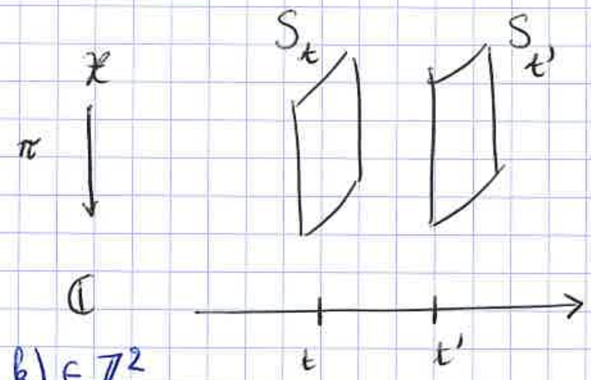
On $U_1 \times W$ we set $\zeta_1 = \zeta$

On $U_2 \times W$ we set $\zeta_2 = \zeta$

The same remains to do with $W' = (V_1 \neq 0)$.

⚠ Notations are misleading.

d) Deformation of Hirzebruch surface



Choose $(m,k) \in \mathbb{Z}^2$

$\mathcal{X}$ is obtained by patching

$\mathbb{C} \times U_1 \times \mathbb{P}^1 \qquad (t_1, z_1, \zeta_1)$

with $\mathbb{C} \times U_2 \times \mathbb{P}^1 \qquad (t_2, z_2, \zeta_2)$

in the following way:

$$\begin{cases} t_1 = t_2 \\ z_1 = \dfrac{1}{z_2} \\ \zeta_1 = z_2^m \zeta_2 + t z_2^k \end{cases}$$

Thus, $S_0 = F_m$.

$\mathcal{X}$ is a threefold, endowed with a proper*, submersive map onto $\mathbb{C}$: hence it is a (**) family of complex surfaces

* fibers are compact + ... see bellow p. 23

(**) compact (Victor)

If $t \neq 0$, we make the following change of coordinates on $S_t := \pi^{-1}(t)$:

- on $U_1 \times \mathbb{P}^1$
$$\begin{cases} z_1' = z_1 \\ \zeta_1' = \dfrac{z_1^k \zeta_1 - t}{t\, \zeta_1} \end{cases}$$

homography $\begin{vmatrix} z_1^k & -t \\ t & 0 \end{vmatrix} = t^2 \neq 0$

- on $U_2 \times \mathbb{P}^1$
$$\begin{cases} z_2' = z_2 \\ \zeta_2' = \dfrac{\zeta_2}{t z_2^{m-k} \zeta_2 + t^2} \end{cases} \qquad \begin{vmatrix} 1 & 0 \\ t z^{m-k} & t^2 \end{vmatrix} = t^2 \neq 0$$

Rmk In this new coordinates, the patching of $U_1 \times \mathbb{P}^1$ with $U_2 \times \mathbb{P}^1$ becomes :

$$z_1' = \frac{1}{z_2'}$$

$$\zeta_1' = \frac{z_1^k\left(z_2^m \zeta_2 + t z_2^k\right) - t}{t\left(z_2^m \zeta_2 + t z_2^k\right)}$$

$$= \frac{z_2^{m-k}\zeta_2}{z_2^k\left(t z^{m-k}\zeta_2 + t^2\right)}$$

$$= z^{m-2k}\zeta_2'$$

Hence, $\begin{cases} S_0 = F_m \\ S_t = F_{m-2k} \quad \text{whenever } t \neq 0 \end{cases}$

cf. differentiable setting , parity

The output is :

cf. p. 16

* The Hirzebruch surfaces F_m and F_ℓ are diffeomorphic iff $m \equiv \ell\,[2]$ (by Ehresmann's theorem).

Proposition If $m \neq \ell$, then F_m and F_ℓ are not biholomorphic.

Proof

We will show that if $m \neq \ell$, then F_m and F_ℓ have non-isomorphic spaces of holomorphic vector fields.

(?) (Victor)

Lemma

The vector fields on $U_z \times \mathbb{P}^1_\zeta$ are of the form

$$g(z)\frac{\partial}{\partial z} + \left(a(z)\zeta^2 + b(z)\zeta + c(z)\right)\frac{\partial}{\partial \zeta}$$, where g, a, b, c are holomorphic.

(very simple) **Proof** $\quad \mathbb{P}^1_\zeta = W \cup W'$

On $U_z \times W_\zeta \simeq \mathbb{C}^2$, the vector fields are of the form

$$f(z,\zeta)\frac{\partial}{\partial z} + g(z,\zeta)\frac{\partial}{\partial \zeta} \qquad f, g \text{ holomorphic}$$

On $U_z \times W'_{\zeta'}$:

$$\tilde{f}(z,\zeta')\frac{\partial}{\partial z} + \tilde{g}(z,\zeta')\frac{\partial}{\partial \zeta'} \qquad \text{idem}$$

The patching is given by

$$\begin{cases} z = z \\ \zeta = \dfrac{1}{\zeta'} \end{cases}$$

Hence $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z}$

$$\frac{\partial}{\partial \zeta'} = \frac{\partial z}{\partial \zeta'}\frac{\partial}{\partial z} + \frac{\partial \zeta}{\partial \zeta'}\frac{\partial}{\partial \zeta} = -\frac{1}{\zeta'^2}\frac{\partial}{\partial \zeta} = -\zeta^2\frac{\partial}{\partial \zeta}$$

$$\tilde{f}(z,\zeta')\frac{\partial}{\partial z} + \tilde{g}(z,\zeta')\frac{\partial}{\partial \zeta'}$$
$$= \underbrace{\tilde{f}\left(z,\frac{1}{\zeta}\right)}_{\text{holomorphic } = f}\frac{\partial}{\partial z} - \underbrace{\tilde{g}\left(z,\frac{1}{\zeta}\right)\zeta^2}_{\text{holom.} = g}\frac{\partial}{\partial \zeta}$$

Hence $\tilde{f}\left(z,\frac{1}{\zeta}\right)$ writes $g(z)$ (see power series expansion)
and $\tilde{g}\left(z,\frac{1}{\zeta}\right)\zeta^2$ writes, writing analytic expansion in the second variable, $a(z)\zeta^2 + b(z)\zeta + c(z)$

$\square$

Back to the proof of the proposition

The vector fields on $\mathcal{U}_1 \times \mathbb{P}^1$ are of the form

$$g(z_1)\frac{\partial}{\partial z_1} + \left(a(z_1)\zeta_1^2 + b(z_1)\zeta_1 + c(z_1)\right)\frac{\partial}{\partial \zeta_1}$$

The vector fields on $\mathcal{U}_2 \times \mathbb{P}^1$ are of the form

$$\tilde{g}(z_2)\frac{\partial}{\partial z} + \left(\tilde{a}(z_2)\zeta_2^2 + \tilde{b}(z_2)\zeta_2 + c(z_2)\right)\frac{\partial}{\partial \zeta_2}$$

The patching is given by:

$$\begin{cases} z_1 = \dfrac{1}{z_2} \\ \zeta_1 = z_2^m \zeta_2 \end{cases}$$

$$\frac{\partial}{\partial z_2} = \frac{\partial z_1}{\partial z_2}\frac{\partial}{\partial z_1} + \frac{\partial \zeta_1}{\partial z_2}\frac{\partial}{\partial \zeta_1} = -z_1^2\frac{\partial}{\partial z_1} + m\underbrace{z_1^{1-m} z_1^m \zeta_1}_{z_1\zeta_1}$$

$$\frac{\partial}{\partial \zeta_2} = z_1^{-m}\frac{\partial}{\partial \zeta_1}$$

$$\tilde{g}(z_2)\frac{\partial}{\partial z_2} + \left(\tilde{a}(z_2)\zeta_2^2 + \tilde{b}(z_2)\zeta_2 + \tilde{c}(z_2)\right)\frac{\partial}{\partial \zeta_2}$$

$$= \tilde{g}\left(\frac{1}{z_1}\right)\left(-z_1^2\frac{\partial}{\partial z_1} + m z_1\zeta_1\frac{\partial}{\partial \zeta_1}\right) + \left(\tilde{a}\left(\frac{1}{z_1}\right)z_1^{m}\zeta_1^2 + \tilde{b}\left(\frac{1}{z_1}\right)\cancel{z_1^m}\zeta_1 + \frac{1}{z_1^m}\tilde{c}\left(\frac{1}{z_1}\right)\right)\frac{\partial}{\partial \zeta_1}$$

$$= \underbrace{-z_1^2\tilde{g}\left(\frac{1}{z_1}\right)}_{\text{holomorphic} = g(z_1)}\frac{\partial}{\partial z_1} + \Bigg[\overbrace{\tilde{a}\left(\frac{1}{z_1}\right)z_1^m}^{\text{holom.}}\zeta_1^2 + \underbrace{\left(m\tilde{g}\left(\frac{1}{z_1}\right)z_1 + \tilde{b}\left(\frac{1}{z_1}\right)\right)}_{\text{holom.}}\zeta_1 + \underbrace{z_1^{-m}\tilde{c}\left(\frac{1}{z_1}\right)}_{\text{holomorphic (set } \zeta_1 = 0)}\Bigg]\frac{\partial}{\partial \zeta_1}$$

Hence $\exists (\alpha, \beta, \gamma) \in \mathbb{C}^3$ s.t. $g(z_1) = \alpha z_1^2 + \beta z_1 + \gamma \longrightarrow \dim 3$

$a(z_1)$ = polynomial of degree at most m in z_1

$\hookrightarrow \dim m+1$

We know that $z_1 \tilde{g}\left(\frac{1}{z_1}\right) = -\frac{\alpha z_1^2 + \beta z_1 + \delta}{z_1^{\cancel{4}}}$

which implies $\tilde{b}\left(\frac{1}{z_2}\right) = \frac{\alpha}{z_1} + \delta$ gives $\xrightarrow{\text{extra}}$ dim 1

$b(z_1) = \delta$

For $m = 0$ $\quad \tilde{c}\cancel{\xi} = \eta \quad$ dim 1

$m \neq 0 \quad \tilde{c} = 0 \quad$ dim 0.

The dimension of the space ~~of vector~~ $\Gamma(F_m, T_{F_m})$ of holomorphic vector fields on F_m is :

see lemma

$$\begin{cases} \dim \Gamma(F_0, T_{F_0}) = 6 & (\text{recall that } F_0 = \mathbb{P}^1 \times \mathbb{P}^1) \\ \dim \Gamma(F_m, T_{F_m}) = m+5 & \text{if } m \neq 0 \end{cases}$$

+ p. 20

If $m \neq \ell$ and $m \not\equiv \ell\ [2]$ then $\dim \Gamma(F_m, T_{F_m}) \neq \Gamma(F_\ell, T_{F_\ell})$.

If $m \neq \ell$ then F_m and F_ℓ are not ~~diffeomorphic~~ biholomorphic. $\square$

Rq Ici, on a calculé à la main les conditions de recollement pour en tirer des conclusions. La théorie générale pour cela est la théorie des faisceaux et la cohomologie.

Properness of $\pi : \mathcal{X} \longrightarrow \mathbb{C}$

By the change of coordinates for an open set $\mathcal{U}$ in $\mathbb{C} - \{0\}$

$\pi|_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is equivalent to a product.

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}) & \xrightarrow[\text{diffeo}]{\text{bi-holom.}} & \mathcal{U}_t \times F_{m-2k} \\ t, z_1, \zeta_1 & & t, z_1, \zeta_1' \\ t, z_2, \zeta_2 & & t, z_2, \zeta_2' \\ \downarrow & & \downarrow \\ \mathcal{U}_t & \longrightarrow & \mathcal{U}_t \end{array} \qquad \begin{array}{l} z_1 = \frac{1}{z_2} \\ \zeta_1' = z_2^{m-2k} \zeta_2' \end{array}$$

this is a proper map, hence the left one is

$\Longrightarrow$ By Ehresmann theorem, the same argument works in the differentiable category for small open sets $\mathcal{U}$ containing 0.

In fact, this doesn't work: Ehresmann requires properness. ~~It~~ We need to consider the explicit equations.